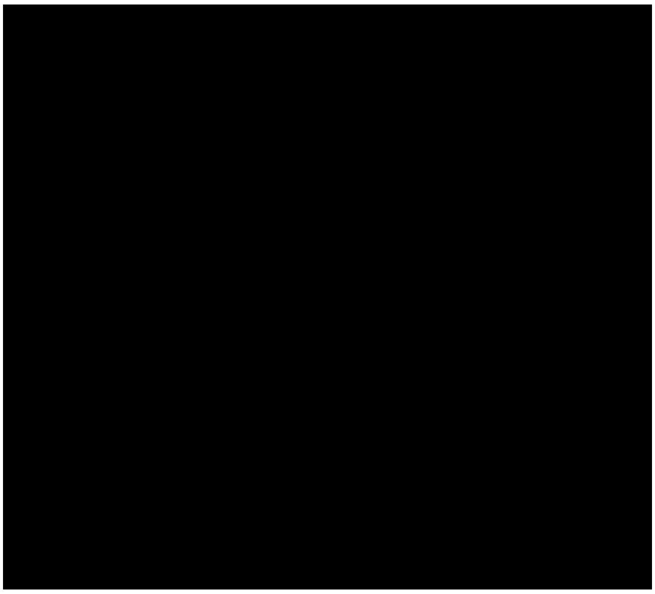
The 4th IOAA
Theoretical Competition
Long Problem

16) A spacecraft is launched from the Earth, it is quickly accelerated to its maximum velocity in the direction of the heliocentric orbit of the Earth, such that its orbit is a parabola with the Sun at its focus point, and grazes the Earth orbit. Take the orbit of the Earth and Mars as circles on the same plane, with radius of $r_E = 1AU$ and $r_M = 1.5AU$, respectively. Make the following approximation: during most of the flight only the gravity from the Sun needs to be considered, but during the brief encounter with a planet, only the gravity of the planet needs to be considered.

Figure 1: 

The trajectory of the spacecraft (not in scale).The inner circle is the orbit of the Earth, the outer circle (red) is the orbit of Mars.

Questions:

(a) What is the angle ψ between the orbit of the spacecraft and the orbit of the Mars (see Fig. 1) as it crosses the orbit of the Mars, without considering the gravity effect of the Mars?

(b) Suppose the Mars happens to be very close to the crossing point at the time of the crossing, from the point of view of an observer on Mars, what is the approaching velocity and direction of approach (with respect to the Sun) of the spacecraft before it is significantly affected by the gravity of the Mars?

Solution:

(a) The orbit of the spacecraft is a parabola, this suggests that the (specific) energy with respect to the Sun is initially

$$\varepsilon = 1/2v_{\max}^2 + U(r_E) = 0$$

and

$$v_{\max} = \sqrt{2U} = \sqrt{2k_{sun}/r_E}$$

The angular momentum is

$$l = r_E v_{\max} = \sqrt{2k_{sun} r_E}$$

When the spacecraft cross the orbit of the Mars at 1.5 AU, its total velocity is

$$v = \sqrt{2U} = \sqrt{2k_{sun} r_M} = \sqrt{\frac{2}{3}} v_{\max}$$

This velocity can be decomposed into v_r and v_θ, using angular momentum decomposition,

$$r_M v_\theta = l = r_E v_{\max}$$

So,

$$v_\theta = \frac{r_E}{r_M} v_{\max} = \frac{2}{3} v_{\max}$$

Thus the angle is given by

$$\cos\psi = \frac{v_\theta}{v} = \sqrt{\frac{r_E}{r_M}} = \sqrt{\frac{2}{3}}$$

or

$$\psi = 32.5^\circ$$

(b) The Mars would be moving on the circular orbit with a velocity

$$v_M \equiv \sqrt{\frac{k_{sun}}{r_M}} = \sqrt{\frac{2}{3}} v_E = 24.32 km/s$$

from the point of view of an observer on Mars, the approaching spacecraft has a velocity of

$$\overrightarrow{v_{rel}} = \vec{v} - \vec{v}_M$$

Now

$$\vec{v} = v\sin\psi\,\hat{r} + v_\theta\,\hat{\theta}$$

with

$$\sin\psi = \sqrt{1-\cos^2\psi} = \frac{1}{\sqrt{3}}$$

So

$$\overrightarrow{v_{rel}} = v\sin\psi\,\hat{r} + (v_\theta - v_M)\,\hat{\theta}$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{2k_{sun}}{r_M}}\,\hat{r} + \left(\frac{2}{3}\sqrt{\frac{2k_{sun}}{r_E}} - \sqrt{\frac{k_{sun}}{r_M}}\right)\hat{\theta}$$

$$= \sqrt{\frac{2k_{sun}}{3r_M}}\,\hat{r} + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{\frac{k_{sun}}{r_M}}\,\hat{\theta}$$

$$= \sqrt{\frac{k_{sun}}{r_M}}(0.8165\,\hat{r} + 0.1547\,\hat{\theta})$$

The approaching velocity is thus

$$v_{rel} = \sqrt{\frac{2}{3} + \left(\frac{2}{\sqrt{3}} - 1\right)^2}\sqrt{\frac{k_{sun}}{r_M}} = 20.21 km/s$$

Remarks: This problem use the so called "patched conics approximation" to solve for spacecraft trajectory when considering gravitional slingshot.

17）The planet Taris

The planet Taris is the home of the Korribian civilization. The Korribian species is a highly intelligent alien life form. They speak Korribianese language. The Korribianese-English dictionary is shown in Table 1; read it carefully! Korriban astronomers have been studying the heavens for thousands of years. Their knowledge can be summarized as follows:

- Taris orbits its host star Sola in a circular orbit, at a distance of 1 Tarislength.
- Taris orbits Sola in 1 Tarisyear.
- The obliquity of Taris is 3° with respect to it's orbit
- There are exactly 10 Tarisdays in 1 Tarisyear.
- Taris has two moons, named Endor and Extor. Both have circular orbits.
- The sidereal orbital period of Endor (around Taris) is exactly 0.2 Tarisdays.
- The sidereal orbital period of Extor (around Taris) is exactly 1.6 Tarisdays.
- The distance between Taris and Endor is 1 Endorlength.
- Corulus, another planet, also orbits Sola in a circular orbit. Corulus has one moon.
- The distance between Sola and Corulus is 9 Tarislengths.

Questions:
(a) Draw the Sola-system, and indicate all planets and moons.
(b) How often does Taris rotate around its axis during one Tarisyear?
(c) What is the distance between Taris and Extor, in Endorlengths?
(d) What is the orbital period of Corulus, in Tarisyears?
(e) What is the distance between Taris and Corulus when Corulus is in opposition?

Korribianese	English translation
Corulus	A planet orbiting Sola
Earth	Name of the third planet orbiting the star "Sun". Scientists claim that there may be life on this planet
Endor	(i) the Goddess of the night; (ii) a moon of Taris
Endorlength	The distance between Taris and Endor
Extor	(i) the God of peace; (ii) a moon of Taris
Mars	Name of the fourth planet orbiting the star "Sun". Scientists claim that there may be life on this planet.
Sola	(i) the God of life; (ii) the star which Taris and Corulus orbit
Sun	Name of a star at a distance of about 250 lightyears from Taris. Scientists discovered that this star has eight planets.
Taris	A planet orbiting the star Sola; home of the Korribians
Tarisday	The time between midnight and midnight on the planet Taris
Tarisyear	The time taken by Taris to make one revolution around Sola
Tarislength	The distance between Sola and Taris

Table 1 - Korribianese-English dictionary

Solution:
(a) The star Sola in the center, with the two planets orbiting Sola. The inner planet Taris has two moons (Endor in the inner orbit, Extor in the outer orbit). The outer planet Corulus has one moon.

(b) There are 10 days-and-nights per Tarisyear. The obliquity is 3 degrees, which means that the planet's rotation is in the same direction as its orbit. The correct answer is therefore 11 rotations per Tarisyear. Note that some students give the incorrect answer of 9 orbits (which is for a counter-rotating planet)

(c) Kepler's law:

$\frac{P^2}{a^3} = const$ So, 4 Endorlengths

(d) Kepler's law:

$$\frac{P^2}{a^3} = const$$ so, 27 Tarisyears

(e) When a planet is in opposition, it is on the opposite side of the Sun, as seen from Taris. So the distance is $9 - 1 = 8$ Tarislengths